BREAKDOWN OF A MONOCHROMATIC WAVE IN A MEDIUM WITH AN INERTIA-FREE NONLINEARITY

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The nonlinear instability mode of a monochromatic wave in a medium with an inertia-free nonlinearity is analyzed theoretically and simulated numerically. It is shown that, if longitudinal and transverse instabilities occur simultaneously, the wave is split into three-dimensional clusters containing amplitude singularities. As a result, the monochromatic wave "breaks down," which is accompanied by a considerable widening of its spectrum and angular divergence.

1. In certain experiments with self-focusing of light [1, 2, 3] one has observed a considerable widening of the originally very narrow spectral line. The purpose of the present study is to explore the mechanism by which the spectrum is widened, specifically in a medium with an inertia-free nonlinearity. Such a mechanism could be the simultaneous buildup of longitudinal and transverse instability in the light wave [4, 5, 6], followed by a split of the wave into three-dimensional clusters which "collapse" within a finite time. This phenomenon may be called the breakdown of a monochromatic wave. The phenomenon is associated not only with light, and it may also be observed in other nonlinear dispersive media; for this reason it seems worthwhile to analyze it from a general point of view.

2. We consider an isotropic nonlinear medium. When the medium is slightly nonlinear, a monochromatic wave

$$\varphi(r,\ t) = \varphi_0 \cos(\omega t - kz), \qquad \omega = \omega_k + q\,|\,\varphi_0\,|^2$$

can travel through it. Here ω_k represents the mode of wave dispersion, and q characterizes the nonlinearity of the medium.

We now consider a nearly monochromatic wave and denote its complex envelope by ψ

$$\varphi(r, t) = \mathrm{Re}\,[\psi(r, t)\exp(-i\omega_k t + ikr)]$$

This envelope satisfies the equation [5, 6, 7, 8, 9]

$$i\left(\frac{\partial\psi}{\partial t} + V_*\frac{\partial\psi}{\partial z}\right) + \frac{\omega_k''}{2}\frac{\partial^2\psi}{\partial z^2} + \frac{V_*}{2k}\Delta_\perp\psi = q\,|\,\psi\,|^2\,\psi \tag{2.1}$$

Here V_* is the group velocity.

For a monochromatic wave we have

$$\psi = \varphi_0 \exp(-iq\,|\,\varphi_0\,|^2 t) \tag{2.2}$$

In an inertia-free dielectric with a scalar nonlinearity mechanism and with the refractive index

$$n = n_0(\omega) + \delta n_*\,|E\,|^2$$

Eq. (2.1) holds true at any elliptical polarization of the wave [10]. Moreover,

$$\omega_k'' = -\frac{1}{V_*^2}\frac{d^2(\omega n_0)}{d\omega^2}, \quad q = -\frac{V_*}{k}\frac{\delta n_*}{n_0}, \quad V_* = \frac{C}{(\omega n_0)_\omega'}$$

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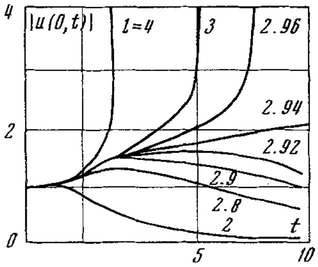

Fig. 1

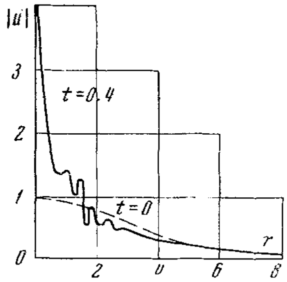

Fig. 2

Linearizing Eq. (2.1) with respect to the solution (2.2) and assuming a perturbation $\delta\psi$

$$\delta\psi \sim \exp\left(-iq\,|\,\varphi_0\,|^2\,t\right) - i\Omega t + ipr$$

we obtain

$$\Omega = V_* p_z \pm \sqrt{q\,|\,\varphi_0\,|^2\left(\omega_k'' p_z^2 + \frac{V_*}{k} p_\perp^2\right) + \frac{1}{4}\left(\omega_k'' p_z^2 + \frac{V_*}{k} p_\perp^2\right)^2} \tag{2.3}$$

If $q > 0$ and $\omega_k'' > 0$, then the monochromatic wave is stable; otherwise, instability will occur. If $q < 0$, there occurs a transverse instability, and self-focusing of the wave results. If at the same time $\omega_k'' > 0$, then there occurs also a longitudinal instability. Only the latter case will be considered further.

We note, in addition, that Eq. (2.1) contains the invariants

$$J_1 = \int |\,\psi\,|^2\, dr$$

$$J_2 = \frac{1}{2}\int\left(\omega_k''\left|\frac{\partial\psi}{\partial z}\right|^2 + \frac{V_*}{k}\,|\,\nabla_\perp\psi|^2 + q\,|\,\psi\,|^4\right) d\mathbf{r}$$

$$\mathbf{p}_\perp = \frac{iV_*}{k}\int(\psi^*\nabla_\perp\psi - \psi\nabla_\perp\psi^*)\, d\mathbf{r} \tag{2.4}$$

$$p_z = i\omega_k''\int\left(\psi^*\frac{\partial\psi}{\partial z} - \psi\frac{\partial\psi^*}{\partial z}\right) d\mathbf{r}$$

3. After changing to dimensionless variables

$$\tau = V_* kt, \quad r_\perp = kr, \quad z' = k\left(\frac{V_*}{k\omega_k''}\right)^{1/2}(z - V_* t)$$

$$u = \left(\frac{|\,q\,|}{2kV_*}\right)^{1/2}\psi$$

we have the equation

$$2i\frac{\partial u}{\partial\tau} + \nabla u + |\,u\,|^2 u = 0 \tag{3.1}$$

with the integrals of motion

$$I_1 = \int |\,u\,|^2\, d\mathbf{r}$$

$$I_2 = \int\left(|\,\nabla u\,|^2 - {}^1/_2\,|\,u\,|^4\right) d\mathbf{r}$$

$$S = i\int(u^*\nabla u - u\nabla u^*)\, d\mathbf{r}$$

Here ∇ is the three-dimensional Laplace operator in the variables $r_\perp$, z'.

We will restrict the analysis to the spherical symmetry case. Then

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}$$

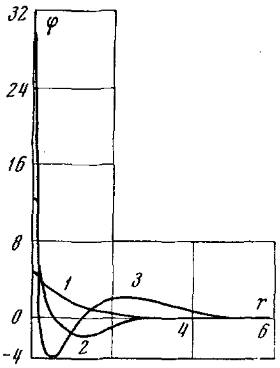

Fig. 3

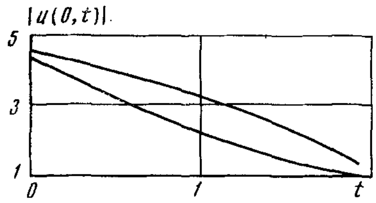

Fig. 4

We introduce into the analysis the following quantities:

$$A = \int_0^\infty r^4 |u|^2 dr$$

$$B = i \int_0^\infty r^3 \left(u^* \frac{\partial u}{\partial r} - u \frac{\partial u^*}{\partial r} \right) dr$$

From Eq. (5) we obtain

$$\frac{\partial A}{\partial t} + B = 0, \qquad \frac{\partial B}{\partial t} + 2I_2 = \frac{1}{2} \int_0^\infty r^2 |u|^4 dr$$

and from this the inequality

$$A < I_2 t^2 + B(0) t + A(0) \tag{3.2}$$

In a medium without nonlinearity, where $q = 0$, the integral J_2 is always positive. In a nonlinear medium there may be $u(r, t)$ distributions for which $J_2 < 0$. We will prove that the development of such distributions will, after a finite time, produce a singularity.

Indeed,

$$I_2 = 0.25q \left(\frac{k}{V_*^3 \omega_k''} \right)^{1/2} J_2$$

and is also negative when $J_2 < 0$. After a finite time, according to (3.2), A should also become negative but, on the other hand, A is a large positive quantity. This contradiction indicates a breakdown of the solution to Eq. (2.4) after a finite time, when $J_2 < 0$.

4. Equation (3.1) was analyzed numerically on the BÉSM-6 computer at the Computation Center, Siberian Branch, Academy of Sciences of the USSR. The initial condition was defined in terms of a Gauss distribution

$$u(r, 0) = a_0 \exp(-r^2 / l^2)$$

and the boundary conditions as

$$\frac{\partial u}{\partial r}(0, t) = 0, \qquad u(\infty, t) = 0$$

Equation (3.1) (with a spherical Laplace operator) was approximated by an implicit difference grid with a variable interval along the radius. The interval at the periphery was made over 1000 times larger than at the center, and this made it feasible to integrate numerically over a sufficiently large radius r (over 20 times larger than the initial half-width of the distribution). The accuracy of computations was checked by how closely the invariance of I_1 and I_2 was maintained. The trend followed by the amplitude $|u(0, t)|$ for $a_0 = 1$ at various values of I_2 is shown in Fig. 1. As can be seen here, at sufficiently small values of I_2 a singularity — a "pull down" — appears after a finite time. As the machine simulated experiment indicates, a negative I_2 is not a necessary condition for a pull-down — it occurs already when $I_2 < 0.675$. A typical pattern — an appearance of the singularity — is shown in Fig. 2 (for $l = 4$). Evidently, the singularity envelops a region of rather small radii.

It was actually possible to observe a pull-down with $|u(0, t)|$ up to approximately 6.3. After that, because of the high gradients near the axis, the integral of motion I_2 broke down. However, integral I_1 at that instant and during the subsequent period of time remained unchanged within $\Delta I_1/I_1 \sim 10^{-5}$. Thus, one may conclude that further computations will be incorrect near the singularity only, and one may reliably continue to follow the evolution of the remainder of the profile.

At $t \to t_0$ there appears a "ripple" across the entire profile, which may be interpreted as the appearance of waves propagating from the central region. These waves carry away the positive value of integral I_2; near the singularity, the intensity of I_2 tends toward $-\infty$.

The steady-state solutions of Eq. (5) in the form $u = e^{it/2}\varphi(r)$ were also obtained numerically. Here

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial \varphi}{\partial r} - \varphi - \varphi^3 = 0, \quad \varphi_r(0) = 0, \quad \varphi(\infty) = 0 \tag{4.1}$$

Equation (4.1) has an infinite number of solutions; graphs of the first three are shown in Fig. 3. The amplitudes of $\varphi(0)$ and the values of integral I_1 for the first set of modes are

	1	2	3	4	5	6	7
$\varphi(0)$	4.34	14.06	28.68	46.87	65.45	81.38	94.39
J_1	1.50	9.47	28.72	64.09	121.02	205.68	324.18

After multiplying Eq. (4.1) by $r^2\varphi$ and integrating over r from 0 to ∞, we have

$$-\int_0^\infty r^2\varphi_r^2 dr - \int_0^\infty r^2\varphi^2 dr + \int_0^\infty r^2\varphi^4 dr = 0$$

After multiplying further by $r^3\varphi_r$ and integrating, we have

$$-\frac{1}{2}\int_0^\infty r^2\varphi_r^2 dr - \frac{3}{2}\int_0^\infty r^2\varphi^2 dr + \frac{3}{2}\int_0^\infty r^2\varphi^4 dr = 0$$

From here

$$I_2 = \int_0^\infty r^2\varphi_r^2 dr - \frac{1}{2}\int_0^\infty r^2\varphi^4 dr = \int_0^\infty r^2\varphi^2 dr = I_1 > 0$$

The integral I_2 is positive and equal to I_1 for all steady-state solutions. This result has been confirmed by very precise computations (5-7 digits) and it proves further that the analysis is correct.

The stability of the first steady-state mode was also analyzed numerically by introducing a Gauss perturbation into the amplitude, adding thus a relative increment $\delta I_1/I_1 \sim 10^{-2}$ to the integral. The steady-state mode dissipated, whether the increment was negative or positive.

The graph in Fig. 4 shows how the magnitude of the amplitude varies with time at $r = 0$.

5. The breakdown of a wave can be interpreted physically as follows. At $\omega_k'' = 0$ a wave beam comprises an aggregate of infinitely thin transverse "layers" not interacting with one another. In every layer there develops a transverse instability splitting it into regions whose dimensions are of the order

$$l_1 \sim \frac{1}{k}\left(\frac{V_*}{k|\psi|^2}\right)^{1/2}$$

In the case of axial symmetry these regions have the shape of annular zones. The regions of rising intensity collapse and form foci; as the pulse shape varies continuously, the foci travel along the z axis (see [11]). The amplitude of field intensity at a focus is limited either by multiphoton absorption or by nonlinearity saturation.

At a finite value of $\omega_k'' > 0$ there occurs interaction between the layers, which results in a redistribution of energy among them. During the first stage, this interaction produces an increasing longitudinal modulation by the characteristic dimension $l_\parallel \sim l_\perp (k\omega_k''/V_*)^{1/2}$.

At the same time there develops a transverse instability in the dimension $l_\perp$ and, in this way, the wave splits into three-dimensional clusters. After a finite time, inside every such cluster there builds up an amplitude singularity of the wave; the region near such a singularity radiates a wide frequency and phase spectrum. The concurrent intense longitudinal modulation of the wave explains the widening of the spectrum.

In the absence of any dissipation mechanisms, the buildup of a singularity continues until its dimension becomes comparable to the wavelength. At that time the spectral line widens most: $\Delta\omega \sim \omega$. In other cases the buildup of a singularity is limited by nonlinearity saturation, by multiquantum absorption, or by the finite relaxation time τ^* of the medium. In the latter case, the line widens by $\Delta\omega \sim 1/\tau^*$ only. When the relaxation time is sufficiently long,

$$\tau^* \gg \frac{1}{\omega}\left(\frac{\omega_k''}{q|\psi|^2}\right)^{1/2} \sim \frac{1}{\omega}\left(\frac{n_0}{\delta n_* |E|^2} \frac{k\omega_k''}{V_*}\right)^{1/2}$$

The pattern of wave breakdown becomes completely "smudged" when the nonlinearity relaxes. We note here that not the entire nonlinearity mechanism need be inertia-free to make it possible for a wave to break down, and that a nonlinearity mechanism with inertia – if present – will produce a wave guide with the wave breaking down inside.

The condition $J_2 < 0$, which is sufficient for the breakdown of a wave, expresses the requirement that the amplitude and the phase of a wave must not change too much within the dimensions $l_\parallel$ and $l_\perp$. This also means that the intensity of a wave beam must be much higher than critical and the period must be much longer than $l_\parallel/V_*$. These requirements are easily met in experiments with laser pulses in nonlinear dielectrics.

Inasmuch as the generated singularities are integrable, only small quantities of the wave energy are "trapped" in them. Nevertheless, the buildup of a longitudinal-transverse instability results in an intensive "turbulization" of the originally monochromatic wave. The characteristic scale dimensions of turbulence are $l_\perp$ in the transverse direction and $l_\parallel$ in the longitudinal direction. The turbulence is strong because within these dimensions the linear terms of Eq. (2.1) are of the same order of magnitude as the nonlinear ones. Turbulization of a wave occurs over the distance $l^* \sim V_*/q|\psi_0|^2$. If a plane-parallel wave beam is injected into the medium, it will transform into a "turbulent jet" along the distance $l \sim l^*$ with a divergence angle $\theta \sim (q|\psi_0|^2/\omega_k)^{1/2}$.

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